DETERMINATION OF A MAGNETIC FIELD WITH ROTATIONAL SYMMETRY ABOUT A GIVEN LINE OF MAGNETIC FORCE

E. F. Afanas'ev and E. A. Morozova

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 7, No. 5, pp. 73–76, 1966

In certain technical applications when a magnetic field with rotational symmetry has to be calculated, it is often a principal requirement that a given line should be a magnetic line of force (or more accurately, that a given surface of rotation should coincide with a surface of magnetic force).

An exact particular solution of the problem is found in the case when the given line of force is straight. This solution is subsequently generalized to the case of an arbitrary smooth line, approximating it by a broken line. A method is also proposed for producing and calculating a magnetic field satisfying the above conditions.

The solution of this problem may be used in questions of magnetohydrodynamics and plasmadynamics as the first approximation for the magnetic field in the case of small magnetic Reynolds numbers, when it is required that a certain line of fluid flow should coincide with a magnetic line of force.

§1. DETERMINATION OF A MAGNETIC FIELD WITH ROTATIONAL SYMMETRY FROM ITS VALUE AT THE AXIS [1]

In the case when there are no currents in the medium, or when they are neglected, a static magnetic field with rotational symmetry is specified by the following formulas in a cylindrical system of coordinates zr

$$H_z(z, r) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} H^{(2n)}(z) \left(\frac{r}{2}\right)^{2n} \qquad (1.1)$$

$$H_r(z, r) = -\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+1)!} H^{(2n+1)}(z) \left(\frac{r}{2}\right)^{2n+1} \qquad (1.2)$$

where H_z and H_r are the component vectors of the magnetic field strength, H(z) is the value of the field at the axis, $H^{(n)}(z)$ is the n-th order derivative.

It is well known that magnetic fields with rotational symmetry are obtained using circular current conductors.

From the Biot-Savart law the magnetic field of a coil on the axis of symmetry is

$$H(z) = \frac{R^2}{2} \int_{z_1}^{z_2} \frac{D(\zeta)\, d\zeta}{[(z-\zeta)^2 + R^2]^{3/2}} \,. \qquad (1.3)$$

Here R is the radius of the coil or solenoid, extending from z_1 to z_2, $D(\zeta)$ is the amp-turn density. Here $D(\zeta) = In(\zeta)$, where I is the current strength and $n(\zeta)$ is the number of turns per unit length.

If the field distribution H(z) is given on the axis, then the amp-turn density $D(\zeta)$ necessary to create such an axial field may be found by solving the integral equation (1.3).

Assuming that the function D(z) is determined for all values of $-\infty < z < +\infty$, the integral equation (1.3) may be solved by means of a Fourier transform when we set $z_1 = -\infty$ and $z_2 = \infty$. The solution as found by V. Glazer has the form [2]

$$D(z) = -\frac{1}{R\pi^2} \int_{-\infty}^{+\infty} \frac{d\xi}{|\xi|\, H_1^{(1)}(iR\,|\xi|)} \int_{-\infty}^{+\infty} e^{i\xi(z-\eta)} H(\eta)\, d\eta \qquad (1.4)$$

where $H_1^{(1)}(x)$ is a Hankel function of the first kind and of the first order, i is the square root of minus one.

The definite integrals in Eq. (1.4) may be evaluated only in some particular cases. Usually it is very difficult to evaluate them even numerically. It is most probably simpler to determine the function D(z) not from Eq. (1.4) but by integrating an equation (1.3) of the first sort numerically.

§2. SOLUTION OF THE PROBLEM WHEN THE GIVEN MAGNETIC LINE OF FORCE IS STRAIGHT

In calculating a magnetic field with rotational symmetry let us impose the requirement that a given curve $r = r_0(z)$ (more precisely, surface of rotation) should coincide with a magnetic line of force. In these cases the magnetic field components H_z and H_r should satisfy the condition

$$\left.\frac{H_r(z, r)}{H_z(z, r)}\right|_{r=r_0(z)} = \frac{dr_0(z)}{dz} \,. \qquad (2.1)$$

We shall consider the case when the line $r = r_0(z)$ is straight:

$$r_0 = kz + a \qquad (kz + a > 0) \,. \qquad (2.2)$$

We shall represent condition (2.1) in the form

$$H_z(z, r_0)\, r_0'(z) - H_r(z, r_0) = 0 \,. \qquad (2.3)$$

Setting H_z and H_r from Eqs. (1.1) and (1.2) into Eq. (2.3), we obtain an equation for the function H(z):

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} r_0^{2n}(z) \times \left[r_0'(z) H^{(2n)}(z) + \frac{r_0(z)}{2(n+1)} H^{(2n+1)}(z) \right] = 0 \,. \qquad (2.4)$$

We shall seek a solution of Eq. (2.4) in the form

$$H(z) = C / r_0^m(z) \qquad (2.5)$$

where C is an arbitrary constant, and the index m is as yet undetermined.

It follows from Eqs. (2.2) and (2.5) that

$$H^{(2n)}(z) = \frac{C}{(m-1)!} \frac{(2n+m-1)!\, k^{2n}}{r_0^{2n+m}(z)} \,, \qquad (2.6)$$

$$H^{(2n+1)}(z) = -\frac{C}{(m-1)!}\frac{(2n+m)!\,k^{2n+1}}{r_0^{2n+m+1}(z)}. \qquad (2.6)\ \text{(cont'd)}$$

Setting Eq. (2.6) into Eq. (2.4), we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n(2n+m-1)!}{2^{2n}(n!)^2}k^{2n+1}\times$$

$$\times\left[1-\frac{2n+m}{2(n+1)}\right]\frac{C}{(m-1)!\,r_0^m(z)} = 0. \qquad (2.7)$$

Dividing both sides of Eq. (2.7) by the factor

$$\frac{C}{(m-1)!\,r_0^m(z)} \neq 0$$

we obtain the relation

$$\sum_{n=0}^{\infty}\frac{(-1)^n(2n+m-1)!}{2^{2n}(n!)^2}\left[1-\frac{2n+m}{2(n+1)}\right]k^{2n+1} = 0. \qquad (2.8)$$

The left side of Eq. (2.8) is a series in powers of the parameter k. This series should sum to zero for any value of k. This is possible if and only if

$$1-\frac{2n+m}{2(n+1)} = 0.$$

It follows from this that the index $m = 2$.

Consequently in the case of the given line of magnetic force of Eq. (2.1), the magnetic field strength on the axis of symmetry is

$$H(z) = \frac{C}{(kz+a)^2}. \qquad (2.9)$$

Setting Eq. (2.9) for H(z) in Eqs. (1.1) and (1.2) we obtain

$$H_z = \frac{C}{(kz+a)^2}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)!}{(n!)^2}\left(\frac{k}{2}\right)^{2n}\left(\frac{r}{kz+a}\right)^{2n} \qquad (2.10)$$

$$H_r = \frac{C}{(kz+a)^2}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+2)!}{n!\,(n+1)!}\left(\frac{k}{2}\right)^{2n+1}\left(\frac{r}{kz+a}\right)^{2n+1}. \qquad (2.11)$$

The absolute value of $H_0(z)$ the magnetic field strength vector on the line $r_0 = kz + a$ is

$$H_0(z) = \sqrt{1+r_0'^2(z)}\,H_z(z, r_0) =$$

$$= \frac{\sqrt{1+k^2}}{(kz+a)^2}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)!}{(n!)^2}\left(\frac{k}{2}\right)^{2n}. \qquad (2.12)$$

The arbitrary constant C appears in formulas (2.9)–(2.12). Any required mean value of the magnetic field strength at any point, and in particular on the line of force $r = kz + a$ may be achieved by appropriate choice of the constant C. Using the criteria for series convergence it is not difficult to establish that the series (2.10) and (2.11) converge absolutely and uniformly with respect to r for

$$r < r^* = \frac{kz+a}{|k|}. \qquad (2.13)$$

It is to be expected that the series (2.10) and (2.11) with alternating signs converge conditionally in a much larger region than in Eq. (2.13). We note that the magnetic field as found in Eqs. (2.10) and (2.11) is not conical since

$$\frac{H_r}{H_z} = \frac{k}{k+a/z}\frac{r}{z} \qquad (a\neq 0,\ r\geqslant a)$$

and tends to a conical form only for $z \to \infty$.

If the magnetic field (2.10), (2.11) is produced by a coil or solenoid of radius $R(R > r_0(z))$ and extent $0 \le z \le l$, then the inverse problem of determining the required density of amp-turns D(z) reduces to the solution of the following integral equation, in accordance with Eq. (1.3)

$$\frac{C}{(kz+a)^2} = \frac{R^2}{2}\int_0^l\frac{D(\zeta)\,d\zeta}{[(z-\zeta)^2+R^2]^{3/2}} \qquad (0\leqslant z\leqslant l) \qquad (2.14)$$

which may be transformed to the simpler form

$$\int_0^1\frac{u(\xi)\,d\xi}{[(x-\xi)^2+\alpha^2]^{3/2}} = \frac{1}{(x+\beta)^2} \qquad (0\leqslant x\leqslant 1)$$

$$x = \frac{z}{l},\quad \alpha = \frac{\beta}{l},\quad \beta = \frac{a}{lk},\quad u(x) = \frac{k^2R^2}{2C}D(lx). \qquad (2.15)$$

Equation (2.15) is an integral equation of the first kind with a kernel which is a function of the modulus of the difference of the arguments, and the finite interval in the change of variables. The theory developed by G. A. Grinberg [3] and the Wiener-Hopf method may be used to solve it. Integral equations of the type (2.15) have been treated in paper [4] for example. Usually the solution of such equations is in the form of a complicated series of multiple definite integrals and is generally found numerically.

§3. SOLUTION OF THE PROBLEM IN THE CASE WHEN THE GIVEN MAGNETIC LINE IS ARBITRARY

Let it be required to create a magnetic field with rotational symmetry such that an arbitrary line (more precisely a surface or rotation) $r = r_0(z)$ is a magnetic line of force.

As regards the function $r = r_0(z)$ we shall assume that it is a single valued function and is positive and bounded in the interval $[0, l]$.

We shall partition interval $0 \le z \le l$ into N equal or unequal intervals $z_{i-1} \le z \le z_i$, where $z_0 = 0$, $z_N = l$, $i = 1, 2, \ldots, N$. Let $r_i = r_0(z_i)$ be a straight segment in each interval, joining the points (z_{i-1}, r_{i-1}) and (z_i, r_i). Consequently the curve $r = r_0(z)$ will be approximated by a broken line of N sections. The results of the preceding paragraph may be used to calculate the magnetic field in each interval. Namely we use the formulas (2.10) and (2.11) for H_z and H_r where for $z_{i-1} \le z \le z_i$ we must set

$$r_0(z) = k_i z + a_i,\quad k = k_i,\quad C = C_i \quad (i = 1, 2, \ldots N)$$

$$k_i = \frac{r_0(z_i) - r_0(z_{i-1})}{z_i - z_{i-1}},$$

$$a_i = \frac{z_i r_0(z_{i-1}) - z_{i-1} r_0(z_i)}{z_i - z_{i-1}} \qquad (i = 1, \ldots, N).$$

In the interval indicated the axial magnetic field, in accordance with Eq. (2.9), is

$$H(z) = \frac{C_i}{(k_i z + a_i)^2} = H_i(z) \quad (z_{i-1} \leqslant z \leqslant z_i). \tag{3.1}$$

The inverse problem of determining the density of amp-turns D(z) may be approached as follows. We take the density D(z) to be approximately constant and equal to D_i in each interval $z_{i-1} < z < z_i$. We then represent the integral in Eq. (2.14) in the form of a sum of integrals over the intervals and carry out the integration. As a result we have

$$\sum_{j=1}^{N} D_j [F_j(z) - F_{j-1}(z)] = H_i(z) \qquad (z_{i-1} < z < z_i) \tag{3.2}$$

$$F_j(z) = \frac{1}{2} \frac{z - z_j}{\sqrt{(z - z_j)^2 + R^2}} \qquad (R > r_0(z)). \tag{3.3}$$

We set $z = (z_{i-1} + z_i)/2 = z_{i-1/2}$ in Eqs. (3.1)–(3.3), and so obtain a system of N linear algebraic equations for $D_1, \ldots, D_N$:

$$\sum_{j=1}^{N} [F_j(z_{i-1/2}) - F_{j-1}(z_{i-1/2})] D_j = H_i(z_{i-1/2})$$

$$(i = 1, 2, \ldots, N). \tag{3.4}$$

If the system of Eq. (3.4) is solved on a computer we obtain the density distribution of amp-turns.

The investigation of this problem was proposed by the recently deceased Evgraf Sergeevich Kuznetsov.

The authors would like to mention that it was due to the constant interest of E. S. Kuznetsov and his valuable advice that the present paper was completed.

REFERENCES

1. A. Rusterholtz, Electron Optics [Russian translation], Izd. inostr. lit., 1952.

2. V. Glazer, The Bases of Electron Optics [Russian translation], Gostekhizdat, 1957.

3. G. A. Grinberg, "Integral equations with a kernel which is a function of the modulus of the difference of the arguments and with a finite interval of change in the variables," Dokl. AN SSSR, vol. 128, no. 3, 1959.

4. E. F. Afanas'ev, "Some problems involving the equation of thermal conductivity with mixed boundary conditions," Differentsial'nye uravneniya [Differential Equations], Minsk, vol. 1, no. 5, 1965.

14 April 1966 Moscow